Enumeration of Rota-Baxter Words[†]

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Abstract

In this paper, we prove results on enumerations of sets of Rota-Baxter words in a finite number of generators and a finite number of unary operators. Rota-Baxter words are words formed by concatenating generators and images of words under Rota-Baxter operators. Under suitable conditions, they form canonical bases of free Rota-Baxter algebras and are studied recently in relation to combinatorics, number theory, renormalization in quantum field theory, and operads. Enumeration of a basis is often a first step to choosing a data representation in implementation. Our method applies some simple ideas from formal languages and compositions (ordered partitions) of an integer. We first settle the case of one generator and one operator where both have exponent 1 (the idempotent case). Some integer sequences related to these sets of Rota-Baxter words are known and connected to other combinatorial sequences, such as the Catalan numbers, and others are new. The recurrences satisfied by the generating series of these sequences prompt us to discover an efficient algorithm to enumerate the canonical basis of certain free Rota-Baxter algebras. More general sets of Rota-Baxter words are enumerated with summation techniques related to compositions of integers.

Key words: Enumerative Combinatorics, Rota-Baxter words and algebras, grammar, Catalan numbers, generating functions, compositions.
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1 Introduction

This paper studies enumeration and algorithms related to generation of sets of Rota-Baxter words which occur naturally as canonical bases of certain Rota-Baxter algebras.

In the 1950s, Spitzer proved a fundamental identity on fluctuation theory in probability by analytic methods. The field of Rota-Baxter algebra was started after G. Baxter (1960) showed that Spitzer's identity follows more generally by a purely algebraic argument for any linear operator P on an algebra satisfying the simple identity

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy) \tag{1}$$

for all elements x, y of the algebra, where λ is a constant.[1] Rota studied this operator through his many articles and communications (see Rota (1995), for example). As a recognition to the great contribution of Rota and Baxter, such a linear operator P is called a **Rota-Baxter operator** and an algebra with the operator is called a **Rota-Baxter algebra**. In spite of diverse applications of Rota-Baxter algebras in mathematics and physics, the study of Rota-Baxter algebra itself has been highly combinatorial. Rota (1969) and Rota and Smith (1972), for instance, related Rota-Baxter operator to other combinatorial identities, such as the Waring formula and Bohnenblust-Spitzer identity. Explicit constructions of free commutative Rota-Baxter algebras have played an important role in further studies, from Cartier (1972) and Rota (1969) in the 1970s, to Guo and Keigher (2000a; 2000b) in the 1990s. Because of the combinatorial nature of the constructions, the related enumeration problems are interesting to study. For example, Guo (2005) showed that free commutative Rota-Baxter algebras on the empty set are related to Stirling numbers of the first and second kind, and these results in general provide generating series for other number sequences.

The unexpected application of non-commutative Rota-Baxter algebras in the work of Connes and Kreimer (2000; 2001) and Ebrahimi-Fard, Guo and Kreimer (2004; 2005) on renormalization of quantum field theory moves the constructions of the corresponding free objects to the forefront. Such constructions were obtained recently by Ebrahimi-Fard and Guo (2004; 2005), providing a

[1] We will take $\lambda = -1$ in this paper. Some authors use $-\theta$ in place of λ.

fuller understanding of the connection first made by Aguiar (see Aguiar (2000) and Ebrahimi-Fard (2002)) between Rota-Baxter algebras and dendriform algebras of Loday and Ronco (2001), and in particular their Hopf algebra of planar trees. Free nonunitary Rota-Baxter algebras are convenient in the study of the adjoint functor from dendriform algebras to Rota-Baxter algebras.

We consider enumeration and algorithmic generation of sets of strings called Rota Baxter words (`RBWs`) which represent expressions in generators and unary operators. Under suitable conditions, these sets form the canonical bases for (non-unitary) free Rota-Baxter algebras with finitely many operators and finitely many generators. These constructions of free Rota-Baxter algebras are only recently explored in special cases in Ebrahimi-Fard and Guo (2004; 2005) and we enumerate not only their canonical bases but also sets that may be useful for more general yet-to-be-explored constructions. For this purpose, we apply concepts and methods from formal languages, grammars, compositions of integers, and generating functions. While most of the sequences and double sequences are found to be closely related to the Catalan numbers, we found sequences that are not covered in the Sloane data base. The enumeration study leads us to an algorithm that generates canonical bases for certain free Rota-Baxter algebras. These algorithmic explorations of enumeration methods (either sequential or randomized) are the first steps that must precede development of important software tools in symbolic computation packages that allow further investigations of algebraic properties of related algebras based on Rota-Baxter words. In addition to Rota-Baxter word sets in several free operators and free generators, we consider sets of `RBWs` where the number of consecutive applications of operators and the number of consecutive generators are bounded. These `RBWs` contain interesting combinatorial structures related to planar rooted trees which form Rota-Baxter algebras as developed in Ebrahimi-Fard and Guo (2005). Some of these properties in certain cases are studied by Aguiar and Moreira (2005). Specifically, Aguiar and Moreira used bijections between combinatorial objects while we applied direct algorithmic enumerations.

The rest of the paper is organized into sections dealing with three levels of generalities with each level built upon the previous one. Section 2 briefly reviews concepts on a free Rota-Baxter algebra and its canonical basis consisting of Rota-Baxter words. Section 3 deals with the case of one idempotent operator and one idempotent generator (that is, exponent 1 case). In Section 4,

we generalize this to arbitrary exponents for one operator and one generator. In Section 5, we further generalize this to arbitrary number of operators and generators, with arbitrary uniform exponents relations. We end with a brief remark on future research.

2 Background and notations

Let $\mathbf{k}$ be a commutative unitary ring and let A be a non-unitary $\mathbf{k}$-algebra. A non-unitary **free Rota-Baxter algebra over** A is a Rota-Baxter algebra $(F(A), P_A)$ together with a $\mathbf{k}$-algebra homomorphism $j_A : A \to F(A)$ with the property that, for any Rota-Baxter algebra (R, P) together with a $\mathbf{k}$-algebra homomorphism $f : A \to R$, there is a unique homomorphism $\overline{f} : F(A) \to R$ of Rota-Baxter algebras such that $f = \overline{f} \circ j_A$.

Ebrahimi-Fard and Guo(2004; 2005) explicitly constructed a free,non-commutative, Rota-Baxter algebra over A, denoted by $\text{Ш}^{\mathrm{NC},0}(A)$, in the case a $\mathbf{k}$-basis X of the $\mathbf{k}$-algebra A exists and is given. Since the enumeration of a $\mathbf{k}$-basis of $\text{Ш}^{\mathrm{NC},0}(A)$ is the main subject of study of this paper, we briefly recall its construction. In what follows, the product of $x_1, x_2 \in X$ in the algebra A is denoted by x_1x_2 or by $x_1 \cdot x_2$ if clarity is needed, and the repeated n-fold product of $x \in X$ in the algebra A is denoted as usual by x^n.

Let $\lfloor$ and $\rfloor$ be symbols, called brackets, and let $X' = X \cup \{\lfloor, \rfloor\}$. Let $S(X')$ be the free (non-commutative) semigroup generated by X'. We can view an element $u \in S(X')$ as a string made up of symbols from X'. The product of two elements $u, v \in S(X')$ is, by an abuse of notation, also denoted by the concatenation uv whenever there is no confusion, and by explicitly using the concatenation operator $\sqcup$ as in $u \sqcup v$ otherwise. It should be emphasized that the operator $\sqcup$ is *not* a symbol in X' and is used solely for the purpose to resolve ambiguity in cases when $u = x_1 \in X$ and $v = x_2 \in X$. We will adopt the convention that the notation $uv = x_1x_2$ always means the product $x_1 \cdot x_2$ in the algebra A (and it may happen that X is closed under algebra multiplication so that $x_1x_2 \in X$), and the concatenation of x_1 with x_2 as elements of $S(X')$ will always be denoted by $x_1 \sqcup x_2$. As we shall see, concatenation of two (or more) elements of X are explicitly excluded in the `RBW` sets, in particular, in any canonical basis of $\text{Ш}^{\mathrm{NC},0}(A)$. So such usage is very limited.

Definition 2.1 A **Rota-Baxter word (**`RBW`**)** w is an element of $S(X')$ that satisfies the following conditions.

(1) The number of $\lfloor$ in w equals the number of $\rfloor$ in w;

(2) Counting from the left to the right, the cumulative number of $\lfloor$ at each location is always greater or equal to that of $\rfloor$;

(3) No subword $x_1 \sqcup x_2$ occurs in w, for any $x_1, x_2 \in X$;

(4) No subword $\rfloor\lfloor$ or $\lfloor\rfloor$ occurs in w;

Interpreting Definition 2.1, a Rota-Baxter word w can be represented uniquely by a finite string composed of one or more elements of X, separated (if more than one x) by a left bracket $\lfloor$ or by a right bracket $\rfloor$, where the set of brackets formed balanced pairs, but neither the string $\rfloor\lfloor$ nor the string $\lfloor\rfloor$ appears as a substring. For example, when $X = \{x\}$, the word $w = \lfloor\lfloor x\rfloor x\lfloor x\rfloor\rfloor x\lfloor x\rfloor$ is an `RBW`, but $\lfloor x \sqcup x\rfloor$, $\lfloor x^2\rfloor$, $\lfloor x\rfloor\lfloor x\rfloor$, $x\rfloor x\lfloor x$, and $\lfloor x\lfloor\,\rfloor x\rfloor$ are not. The number of balanced pairs of brackets in an `RBW` is called its **degree**. The degree of w in the above example is 4.

Let $\mathfrak{M}^0(X)$ be the set of Rota-Baxter words and let $\mathfrak{M}^1(X)$ be $\mathfrak{M}^0(X)$ with the empty (or trivial) word $\emptyset$ adjoined.

Example 2.2 Let $\mathbf{k}$ be a field. Let $A = \mathbf{k}[x]$ be the polynomial ring in one indeterminate x over $\mathbf{k}$. Then $X = \{x^n \mid n \in \mathbb{N}\}$ is a $\mathbf{k}$-basis. In this case, if $a + b = n$, then $\lfloor x^n\rfloor = \lfloor x^a x^b\rfloor$ is an RBW, but $\lfloor x^a \sqcup x^b\rfloor$ is not.

Example 2.3 Let $B = \mathbf{k}[x]/\mathfrak{a}$ be the quotient ring of A of Example 2.2 by the ideal $\mathfrak{a}$ generated by $x^2 - x$. Then writing $\overline{1} = 1 + \mathfrak{a}$ and $\overline{x} = x + \mathfrak{a}$, the set $X = \{\overline{1}, \overline{x}\}$ is a $\mathbf{k}$-basis of B. Here $\lfloor\overline{1}\rfloor$, $\lfloor\overline{1}\cdot\overline{x}\rfloor$, and $\overline{1}\lfloor\overline{x}\rfloor$ are `RBWs` but $\lfloor\overline{1}\sqcup\overline{x}\rfloor$ is not.

Let $Ш^{\mathrm{NC},0}(A)$ be the free $\mathbf{k}$-module with basis $\mathfrak{M}^0(X)$. Ebrahimi-Fard and Guo (2004; 2005) show that the following properties

$$\begin{aligned}
x \diamond x' &= x \cdot x' \\
x \diamond \lfloor w\rfloor &= x\lfloor w\rfloor \\
\lfloor w\rfloor \diamond x &= \lfloor w\rfloor x \\
\lfloor w\rfloor \diamond \lfloor w'\rfloor &= \lfloor\lfloor w\rfloor \diamond w'\rfloor + \lfloor w \diamond \lfloor w'\rfloor\rfloor + \lambda\lfloor w \diamond w'\rfloor
\end{aligned} \tag{2}$$

for all $x, x' \in X$ and all $w, w' \in \mathfrak{M}^0(X)$ uniquely define an associative bilinear

product $\diamond$ on $\text{Ш}^{\mathrm{N}C,0}(A)$. This product, together with the linear operator

$$P_A : \text{Ш}^{\mathrm{N}C,0}(A) \to \text{Ш}^{\mathrm{N}C,0}(A), \qquad P_A(w) = \lfloor w \rfloor \text{ if } w \in \mathfrak{M}^0(X), \tag{3}$$

and the natural embedding

$$j_A : A \to \text{Ш}^{\mathrm{N}C,0}(A), \qquad j_A(x) = x \text{ if } x \in X,$$

makes $\text{Ш}^{\mathrm{N}C,0}(A)$ the free (non-unitary) Rota-Baxter algebra over A. We will not need to know the explicit construction of this associative bilinear product in $\text{Ш}^{\mathrm{N}C,0}(A)$ for the rest of the paper. However, we note that as an element of the algebra $\text{Ш}^{\mathrm{N}C,0}(A)$, the string $\lfloor w \rfloor$ may be interpreted as the image of the operator P_A on w for any $w \in \mathfrak{M}^0(X)$ and that for any such w writable as the concatenation uv for $u, v \in \mathfrak{M}^0(X)$, the concatenation can be viewed as $u \diamond v$ (the first three cases of Eq. (2)) and this justifies the abuse of notation and convention in using concatenation for both the algebra multipication in A and the semigroup product in $S(X')$.

Example 2.4 Let $\mathbf{k}$ be a field. Let $A = \mathbf{k}\langle x_1, \ldots, x_q \rangle$ be the polynomial ring in q non-commutating indeterminates $x_1, \ldots, x_q$ over $\mathbf{k}$. Then the set X of (non-commutative) monomials is a $\mathbf{k}$-basis of A. If $q \geqslant 2$, then $\lfloor x_2^3 x_1^4 x_2^2 \rfloor = \lfloor x_2^3 \cdot x_1^4 \cdot x_2^2 \rfloor$ is an RBW, but $\lfloor x_2^3 \sqcup x_1^4 \sqcup x_2^2 \rfloor$ is not. Moreover, in $\text{Ш}^{\mathrm{N}C,0}(A)$, we have $1 \diamond x_i = x_i = x_i \diamond 1$ and $1 \diamond \lfloor w \rfloor = 1 \lfloor w \rfloor$ for any $w \in S(X')$.

Definition 2.5 The free (non-unitary) Rota-Baxter algebra $\text{Ш}^{\mathrm{N}C,0}(A)$ of Example 2.4 will be denoted by $\text{Ш}^{\mathrm{N}C,0}(q)$ and referred to as the free (non-unitary) Rota-Baxter algebra on q generators $x_1, \ldots, x_q$. The corresponding $\mathbf{k}$-basis $\mathfrak{M}^0(A)$ consisting of `RBWs` built from X will be denoted by $\mathfrak{M}^0(q)$. Any non-commutative monomial $x \in X$ will be called an x-**run**. For any `RBW` $w \in \mathfrak{M}^0(q)$, the **arity** of w is the number of $x_1, \ldots, x_q$ appearing in w, counted with multiplicities. For example, the arity of an x-run is the total degree of the monomial it represents and the `RBW` $w = x_1 x_2^2 \lfloor x_2^3 x_1^4 x_2^2 \rfloor$ has two x-runs and arity 12.

Example 2.6 Let $A = \mathbf{k}\langle x_1, \ldots, x_q \rangle$ be as in Example 2.4, let $\vec{v} = (v_1, \ldots, v_q)$ be a vector of q positive integers, and $\mathfrak{a}$ be the bilateral ideal of A generated by the polynomials $x_i^{v_i+1} - x_i$, $1 \leqslant i \leqslant q$. Let B be the quotient $\mathbf{k}$-algebra $A/\mathfrak{a}$. Writing $\overline{1} = 1 + \mathfrak{a}$ and $\overline{x}_i = x_i + \mathfrak{a}$, let X be the set consisting of all

non-commutative finite power products

$$x = \overline{x}_{j_1}^{e_{j_1}} \cdots \overline{x}_{j_{\ell-1}}^{e_{j_{\ell-1}}} \overline{x}_{j_\ell}^{e_{j_\ell}} \overline{x}_{j_{\ell+1}}^{e_{j_{\ell+1}}} \cdots \overline{x}_{j_r}^{e_{j_r}}$$

in $\overline{x}_i$ $(1 \leqslant i \leqslant q)$, where the indices satisfy $j_{\ell-1} \neq j_\ell$ and $j_\ell \neq j_{\ell+1}$ for all ℓ $(2 \leqslant \ell \leqslant r-1)$, and the exponents satisfy $1 \leqslant e_{j_\ell} \leqslant v_{j_\ell}$ for ℓ $(1 \leqslant \ell \leqslant r)$. Then X is a $\mathbf{k}$-basis of B and $Ш^{\mathrm{NC},0}(B)$ is a free Rota-Baxter algebra on B.

Definition 2.7 The free (non-unitary) Rota-Baxter algebra $Ш^{\mathrm{NC},0}(B)$ of Example 2.6 will be denoted by $Ш^{\mathrm{NC},0}(q, \vec{v})$ and be referred to as the free (non-unitary) Rota-Baxter algebra on q generators $\overline{x}_1, \ldots, \overline{x}_q$ with exponent vector $\vec{v}$. The corresponding $\mathbf{k}$-basis $\mathfrak{M}^0(B)$ consisting of `RBWs` built from X will be denoted by $\mathfrak{M}^0(q, \vec{v})$. Any non-commutative monomial $x \in X$ will be called an x-**run**. For any `RBW` $w \in \mathfrak{M}^0(q, \vec{v})$, the **arity** of w is the number of $\overline{x}_1, \ldots, \overline{x}_q$ appearing in a canonical representation of w as an element of B, counted with multiplicities. For example, the arity of an x-run is the total degree of the monomial it represents and the `RBW` $w = \overline{x}_1\overline{x}_2^2\lfloor \overline{x}_2^3\overline{x}_1^4\overline{x}_2^2 \rfloor$ has two x-runs and arity 12, provided $v_1 \geqslant 4$ and $v_2 \geqslant 3$.

In this paper, we will enumerate `RBWs` in $\mathfrak{M}^0(q, \vec{v})$ (actually $\mathfrak{M}^1(q, \vec{v})$, after adjoining the empty `RBW` $\emptyset$) with a given degree and arity, by giving algorithms to generate them and generating functions that count them. We begin with $q = 1$ and $v_1 = 1$ in Section 3 under some extra hypothesis by restricting to a subset of `RBWs`, but generalize the results to arbitrary q and `RBWs` involving multiple unary operators. For some of these generalizations, we note that the corresponding free Rota-Baxter algebras have not been constructed and the enumeration of the sets of `RBWs` is included for possible future applications.

3 One idempotent operator and one idempotent generator case

In this section, we restrict ourselves to Example 2.6 when $q = 1$ and $\vec{v} = (1)$, that is, $\overline{x}$ is idempotent, and we further assume that the Rota-Baxter operator P is also idempotent (that is, $P(P(w)) = P(w)$ for all w). These restrictions allow us to first focus on the word structures of free Rota-Baxter algebra constructions before considering other factors involved in more general Rota-Baxter words. Interestingly, in most applications of Rota-Baxter algebra in quantum field theory, the operators are idempotent.

We contribute three results for the enumeration of a canonical basis of the free Rota-Baxter algebra in this special case. After reviewing some preliminary material and setting up notations, we consider generating functions based on the degree of the Rota-Baxter words in Section 3.1. In Section 3.2, we refine the study to consider generating functions based on the degree and arity. In Sections 3.3, we give an algorithm to generate this canonical basis with given degree and arity.

For simplicity, we will drop the bar notation above the generator x. Under our current hypothesis that both the single generator x and the operator P are idempotent, let $R = R_{1,1}$ be the subset of $\mathfrak{M}^1(X)$ consisting of $\emptyset$ and Rota-Baxter words w composed of x's and pairs of balanced brackets such that no two x's are adjacent, and no two pairs of brackets can be immediately adjacent or nested. In other words, the strings $\rfloor\lfloor$, $\lfloor\rfloor$, and strings of the form $\lfloor\lfloor * \rfloor\rfloor$ where the brackets are balanced pairs and where $*$ may be any `RBW`, do not appear as substrings of w. For example, the `RBW` $\lfloor\lfloor x\rfloor\rfloor x$ is not element of R. For the rest of this section, all `RBWs` are assumed to be in R.

3.1 Generating functions of one variable

Let $R(n)$ be the subset of R of degree n (with our convention, $R(0) = \{\emptyset, x\}$). For $n > 0$, let $B(n)$ be the subset of $R(n)$ consisting of `RBWs` that begin with a left bracket and end with a right bracket. Words in $B(n)$ are said to be **bracketed**. By pre- or post- concatenating a bracketed `RBW` w with x, we get three new `RBWs`: xw, wx, and xwx, which are called respectively the **left**, **right**, and **bilateral associate** of w. We also consider x to be an associate of the trivial word $\emptyset$. Any non-trivial `RBW` is either bracketed or an associate. Thus for $n > 0$, the set $A(n)$ of all associates form the complement of $B(n)$ in $R(n)$ and it is the disjoint union of these cosets:

$$A(n) = xB(n) \cup B(n)x \cup xB(n)x, \qquad n > 0.$$

The set of bracketed `RBWs` is further divided into two disjoint subsets. The first subset $I(n)$ consists of all **indecomposable** bracketed `RBWs`, whose beginning left bracket and ending right bracket are paired. The second subset $D(n)$ consists of all **decomposable** bracketed `RBWs` whose beginning left bracket and ending right bracket are not paired. For convenience in counting, we define $B(0)$, $I(0)$, $D(0)$ to be the empty set and note that $A(0) = \{x\}$.

The following table lists these various types of `RBWs` in lower degrees.

deg	$I(n)$	$D(n)$	$A(n)$	$B(n)$
0			x	
1	$\lfloor x \rfloor$		$x\lfloor x \rfloor, \lfloor x \rfloor x, x\lfloor x \rfloor x$	$\lfloor x \rfloor$
2	$\lfloor x\lfloor x \rfloor\rfloor, \lfloor\lfloor x \rfloor x\rfloor, \lfloor x\lfloor x \rfloor x\rfloor$	$\lfloor x \rfloor x \lfloor x \rfloor$	12 associates	$I(2) \cup D(2)$

In terms of formal languages, we start with an alphabet Σ of **terminals** consisting of a special symbol $\emptyset$ and the three symbols $\lfloor$, x, and $\rfloor$, a set of **non-terminals** consisting of $\langle\texttt{bracketed}\rangle$, $\langle\texttt{indecomposable}\rangle$, $\langle\texttt{decomposable}\rangle$, $\langle\texttt{associate}\rangle$ and the sentence symbol $\langle\texttt{RBW}\rangle$. Let the production rules be:

$$\langle\texttt{RBW}\rangle \to \emptyset \mid \langle\texttt{bracketed}\rangle \mid \langle\texttt{associate}\rangle \tag{4}$$
$$\langle\texttt{associate}\rangle \to x \mid x\langle\texttt{bracketed}\rangle \mid \langle\texttt{bracketed}\rangle x \mid x\langle\texttt{bracketed}\rangle x \tag{5}$$
$$\langle\texttt{bracketed}\rangle \to \langle\texttt{indecomposable}\rangle \mid \langle\texttt{decomposable}\rangle \tag{6}$$
$$\langle\texttt{indecomposable}\rangle \to \lfloor \langle\texttt{decomposable}\rangle \rfloor \mid \lfloor \langle\texttt{associate}\rangle \rfloor \tag{7}$$
$$\langle\texttt{decomposable}\rangle \to \langle\texttt{bracketed}\rangle x \langle\texttt{bracketed}\rangle \tag{8}$$

By (5) and (7), it is clear that the sentences in this language will be `RBWs` and vice versa. The production rules thus define a **grammar** whose **language** will be $\bigcup_{n=0}^{\infty} R(n)$. For the benefit of readers unfamiliar with this area of computer science, we refer them to Aho and Ullman (1972) for basics and now give a detailed proof.

Proof. Let $\mathfrak{L}$ be the language defined by the grammar above. Clearly, $R(0) \subset \mathfrak{L}$. It is easy to see by induction on the length of the string representing an `RBW` w that $w \in \mathfrak{L}$ in case w is either indecomposable or an associate since by Definition 2.1, removing the outermost pair of balanced brackets from an indecomposable `RBW` or the appended x (or x's) from an associate will yield an `RBW` of shorter length. If w is bracketed and decomposable, then we may write $w = \lfloor w' \rfloor$ where the beginning and ending brackets are not paired and $w' \in S(X')$. Let u be left subword of w ending in and including the $\rfloor$ that balances the beginning $\lfloor$ of w. Then u is an indecomposable `RBW` by definitions. Again by Definition 2.1, the next symbol in w following u must be x, which is then followed by a $\lfloor$ matching the ending $\rfloor$. Thus we may write $w = uxv$ where u, v are both indecomposable (and bracketed) `RBWs` of shorter lengths and by induction, u, v, and hence also w (by Rule (8)), are in $\mathfrak{L}$.

Conversely, to show that $\mathcal{L} \subseteq R$, it is only necessary to show that a sentence production π beginning with the rule $\langle\texttt{RBW}\rangle \to \langle\texttt{bracketed}\rangle$ will result in a string w that is a bracketed RBW (by Rules (4), (5), and Definition 2.1). The shortest sentence production π with this property is

$$\langle\texttt{RBW}\rangle \to \langle\texttt{bracketed}\rangle \to \langle\texttt{indecomposable}\rangle \to \lfloor\langle\texttt{associate}\rangle\rfloor \to \lfloor x\rfloor$$

which results in an RBW. In general, we proceed by induction on the number of productions applied in π. The possible continuations of $\langle\texttt{RBW}\rangle \to \langle\texttt{bracketed}\rangle$ are those leading to $\langle\texttt{decomposable}\rangle$, $\lfloor\langle\texttt{decomposable}\rangle\rfloor$, or $\lfloor\langle\texttt{associate}\rangle\rfloor$, each of which in turns leads back to $\langle\texttt{bracketed}\rangle$. Since application of any of the Rules (5), (7), or (8) will result in an RBW if the non-terminals involved result in RBWs, the induction hypothesis and Definition 2.1 show that these continuations will result in a valid RBW. $\square$

Because $A(n)$ and $B(n)$ are disjoint, and $I(n)$ and $D(n)$ are disjoint, the proof shows that given any RBW w, there is a unique sentence production π ending in w. Sentences derivable from $\langle\texttt{bracketed}\rangle$, $\langle\texttt{indecomposable}\rangle$, $\langle\texttt{decomposable}\rangle$, and $\langle\texttt{associate}\rangle$ correspond bijectively respectively to RBWs that are bracketed, indecomposable, decomposable, and associate.

For $n \geqslant 0$, let r_n (resp. a_n, resp. b_n, resp. i_n, resp. d_n) be the number of all (resp. associate, resp. bracketed, resp. indecomposable, resp. decomposable) RBWs with n pairs of (balanced) brackets. The first few values of r_n for $n = 0, 1, 2, \ldots, 5$ are

$$2, 4, 16, 80, 448, 2688, \ldots$$

which suggests that it is the sequence A025225 from Sloane *et al.* (2005) whose n-th term is given by $2^{n+1}C_n$. Here $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. By (4) and (5), we clearly have $r_n = 4b_n$ for $n > 0$, and hence it suffices to prove that $b_n = 2^{n-1}C_n$. It is known that the sequence A003645 whose n-th term is $2^{n-1}C_n$, ($n \geqslant 1$), has a generating series given by

$$\sum_{n=1}^{\infty} 2^{n-1}C_n z^n = \frac{1 - 4z - \sqrt{1-8z}}{8z}. \tag{9}$$

We will prove that b_n indeed has a generating series given by Eq. 9, and more.

Theorem 3.1 *The generating series for r_n, b_n, i_n, d_n and a_n are given by:*

$$\begin{aligned}
\mathbf{R}(z) &= \sum_{n=0}^{\infty} r_n z^n = \frac{1-\sqrt{1-8z}}{2z}, \\
\mathbf{B}(z) &= \sum_{n=0}^{\infty} b_n z^n = \frac{1-4z-\sqrt{1-8z}}{8z}, \\
\mathbf{I}(z) &= \sum_{n=0}^{\infty} i_n z^n = \frac{1-2z-\sqrt{1-8z}}{2(z+1)}, \\
\mathbf{D}(z) &= \sum_{n=0}^{\infty} d_n z^n = \frac{1-7z+4z^2+(3z-1)\sqrt{1-8z}}{8z(z+1)}, \\
\mathbf{A}(z) &= \sum_{n=0}^{\infty} a_n z^n = \frac{3-4z-3\sqrt{1-8z}}{8z}.
\end{aligned} \tag{10}$$

Corollary 3.2 *The number of Rota-Baxter words of degree n in the canonical basis of the free Rota-Baxter algebra with a single idempotent generator and idempotent operator is given by*

$$r_n = 2^{n+1} C_n, \qquad n = 0, 1, 2, \ldots$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Proof: We note the initial values of the sequences are given by:

$$\begin{aligned}
r_0 &= 2, \quad b_0 = 0, \quad i_0 = 0, \quad d_0 = 0, \quad a_0 = 1, \\
r_1 &= 4, \quad b_1 = 1, \quad i_1 = 1, \quad d_1 = 0, \quad a_1 = 3.
\end{aligned}$$

Thus, it does not matter whether the generating series for $\mathbf{B}(z)$, $\mathbf{I}(z)$, and $\mathbf{D}(z)$ start at $n = 0$ or $n = 1$. By production rule (6), for all $n > 0$, we have $b_n = i_n + d_n$. By production rule (7), an `RBW` w is in $I(n)$ if and only if $w = \lfloor w' \rfloor$ for some $w' \in A(n)$ or $w' \in D(n)$. Thus $i_n = 3b_{n-1} + d_{n-1}$ and it follows that $i_n = 4b_{n-1} - i_{n-1}$ or

$$i_n + i_{n-1} = 4b_{n-1}.$$

Multiply both sides by z^n and sum over $n \geqslant 2$. The left hand side gives

$$\sum_{n=2}^{\infty} i_n z^n + \sum_{n=2}^{\infty} i_{n-1} z^n = \mathbf{I}(z) - z + z\,\mathbf{I}(z).$$

For the right hand side, note that by production rules (7) and (6), every bracketed, decomposable `RBW` is formed from bracketed, indecomposable `RBWs` with x inserted between them. This decomposition is clearly unique and the

degree of the decomposable RBW is the sum of the component indecomposable ones. Hence we have

$$d_n = \sum_{\substack{(n_1,\ldots,n_p;\, n),\\ p>1}} i_{n_1} \cdots i_{n_p} \tag{11}$$

where the notation $(n_1, \ldots, n_p;\, n)$ denotes all compositions $n_1 + \cdots + n_p = n$ of n into p positive integers, and the sum is over all compositions for all lengths $p > 1$. Therefore, noting the special case $p = 1$ corresponds to a single summand i_n, we have

$$b_n = i_n + d_n = \sum_{\substack{(n_1,\ldots,n_p;\, n),\\ p\geqslant 1}} i_{n_1} \cdots i_{n_p}\,, \quad (n \geqslant 1) \tag{12}$$

From this, we obtain

$$\begin{aligned}
\mathbf{B}(z) &= \sum_{n=1}^{\infty} b_n z^n \\
&= \sum_{n=1}^{\infty} \left(\sum_{(n_1,\ldots,n_p;\, n),\, p\geqslant 1} i_{n_1} \cdots i_{n_p} \right) z^n \\
&= \sum_{p=1}^{\infty} \left(\sum_{k=1}^{\infty} i_k z^k \right)^p,
\end{aligned}$$

$$\mathbf{B}(z) = \frac{\mathbf{I}(z)}{1 - \mathbf{I}(z)}\,. \tag{13}$$

Since

$$4 \sum_{n=2}^{\infty} b_{n-1} z^n = 4z\, \mathbf{B}(z)$$

we have

$$\mathbf{I}(z) - z + z\, \mathbf{I}(z) = \frac{4z\, \mathbf{I}(z)}{1 - \mathbf{I}(z)}.$$

Solving for $\mathbf{I}(z)$ with the initial condition $i_0 = 0$, we have

$$\mathbf{I}(z) = \frac{1 - 2z - \sqrt{1 - 8z}}{2(z+1)}.$$

This proves the third formula of Eq. (10) and by Eq. (13), after rationalizing the denominator, we obtain the second formula of Eq. (10). The other formulas are easily derived from the relations $\mathbf{D}(z) = \mathbf{B}(z) - \mathbf{I}(z)$, $\mathbf{A}(z) = 1 + 3\mathbf{B}(z)$, and $\mathbf{R}(z) = 1 + \mathbf{B}(z) + \mathbf{A}(z)$. The corollary comes from the remarks before the theorem. □

Theorem 3.2 shows that the number i_n of bracketed indecomposable `RBWs` of degree n ($n \geqslant 1$) is the n-th term of the sequence A062992:

$$1, 3, 13, 67, 381, 2307, \ldots$$

and the number d_n of bracketed decomposable `RBWs` of degree n ($n \geqslant 2$) is a new sequence A115194, which starts with

$$1, 7, 45, 291, 1917, 12867, \ldots.$$

3.2 Generating functions of two variables

In our computational experiment, we observed that the set $B(n)$, when stratified by the number of x's appearing in an `RBW`, possesses certain very nice properties that may give better combinatorial understanding of how the canonical basis is constructed recursively (see the algorithm in the next subsection). To describe the stratification, for any `RBW` w, recall (Definition 2.7) that w has **arity** m if the number of x's appearing in the string representation of w is exactly m. For any $m \geqslant 0$, let $R(n,m)$ be the subset of R of degree n and arity m, and define similarly the notations $A(n,m)$, $B(n,m)$, $I(n,m)$, and $D(n,m)$. These are all finite sets. Let their sizes be respectively denoted by $r_{n,m}$, $a_{n,m}$, $b_{n,m}$, $i_{n,m}$, and $d_{n,m}$. For initial values, we have

$$\begin{array}{ll}
r_{0,0} = 1; & a_{0,0} = b_{0,0} = i_{0,0} = d_{0,0} = 0; \\
r_{0,1} = a_{0,1} = 1; & b_{0,1} = i_{0,1} = d_{0,1} = 0; \\
r_{1,1} = b_{1,1} = i_{1,1} = 1; & a_{1,1} = d_{1,1} = 0; \\
r_{1,2} = a_{1,2} = 2; & b_{1,2} = i_{1,2} = d_{1,2} = 0; \\
r_{1,3} = a_{1,3} = 1; & b_{1,3} = i_{1,3} = d_{1,3} = 0;
\end{array}$$

$$\begin{array}{rl}
r_{0,m} = a_{0,m} = b_{0,m} = i_{0,m} = d_{0,m} = 0 & \text{for } m \geqslant 2; \\
r_{1,m} = a_{1,m} = b_{1,m} = i_{1,m} = d_{1,m} = 0 & \text{for } m \geqslant 4; \\
r_{n,0} = a_{n,0} = b_{n,0} = i_{n,0} = d_{n,0} = 0 & \text{for } n \geqslant 1; \\
r_{n,1} = a_{n,1} = b_{n,1} = i_{n,1} = d_{n,1} = 0 & \text{for } n \geqslant 2.
\end{array}$$

From the production rules (4) – (8), we see that for $n \geqslant 1, m \geqslant 2$:

$$r_{n,m} = b_{n,m} + a_{n,m} \tag{14}$$

$$a_{n,m} = 2\, b_{n,m-1} + b_{n,m-2} \tag{15}$$

$$b_{n,m} = i_{n,m} + d_{n,m} \tag{16}$$

$$i_{n,m} = d_{n-1,m} + a_{n-1,m} \tag{17}$$

Now for $n \geqslant 2, m \geqslant 2$ and any $w \in D(n,m)$, we can write w uniquely as $w_{n_1} x w_{n_2} \cdots x w_{n_p}$ where $w_{n_j} \in I(n_j)$ and $n_1 + \cdots + n_p$ is a composition of n using p positive integers. Let m_j be the arity of w_{n_j}. Then clearly, $m_1 + \cdots + m_p = m - p + 1$ and so we may refine Eq. (11) to:

$$d_{n,m} = \sum_{p=2}^{\min(n,m)} \sum_{(m_1,\ldots,m_p;\, m-p+1)} \sum_{(n_1,\ldots,n_p;\, n)} (i_{n_1,m_1}) \cdots (i_{n_p,m_p}), \tag{18}$$

and noting that the case $p = 1$ corresponds to a single summand $i_{n,m}$, refine Eqn. (12) to:

$$b_{n,m} = i_{n,m} + d_{n,m} = \sum_{p=1}^{\min(n,m)} \sum_{(m_1,\ldots,m_p;\, m-p+1)} \sum_{(n_1,\ldots,n_p;\, n)} (i_{n_1,m_1}) \cdots (i_{n_p,m_p}) \,. \tag{19}$$

Now from Eq. (15)–(17), we have

$$\begin{aligned} i_{n,m} &= d_{n-1,m} + a_{n-1,m} \\ &= b_{n-1,m} - i_{n-1,m} + 2\, b_{n-1,m-1} + b_{n-1,m-2}\,, \end{aligned}$$

and so

$$i_{n,m} + i_{n-1,m} = b_{n-1,m} + 2\, b_{n-1,m-1} + b_{n-1,m-2}\,. \tag{20}$$

Define the bivariate generating series

$$\mathbf{R}(z,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n,m} z^n t^m$$

and similarly define $\mathbf{B}(z,t)$, $\mathbf{I}(z,t), \mathbf{D}(z,t)$, and $\mathbf{A}(z,t)$. Note that for $\mathbf{B}(z,t)$, $\mathbf{I}(z,t), and \mathbf{D}(z,t)$, it does not matter whether the series indices n, m start at 0 or 1. Multiply both sides of Eq. (20) by $z^n t^m$ and summing up for $n \geqslant 2, m \geqslant 2$, the left hand side gives:

$$\begin{aligned} &\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} i_{n,m} z^n t^m + \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} i_{n-1,m} z^n t^m \\ &= \mathbf{I}(z,t) - zt + z \sum_{n=1}^{\infty} \left(- i_{n,1} z^n t + \sum_{m=1}^{\infty} i_{n,m} z^n t^m \right) \\ &= \mathbf{I}(z,t) - zt - z^2 t + z \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} i_{n,m} z^n t^m \\ &= \mathbf{I}(z,t) - zt - z^2 t + z\, \mathbf{I}(z,t). \end{aligned}$$

Now, we sum the right hand side of Eq. (20) one term at a time.

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} b_{n-1,m} z^n t^m &= z\sum_{n=1}^{\infty}\sum_{m=2}^{\infty} b_{n,m} z^n t^m \\ &= z\,\mathbf{B}(z,t) - z^2 t\end{aligned}$$

$$\begin{aligned}2\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} b_{n-1,m-1} z^n t^m &= 2zt\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} z^n t^m \\ &= 2zt\,\mathbf{B}(z,t)\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} b_{n-1,m-2} z^n t^m &= zt^2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{n,m} z^n t^m \\ &= zt^2\Big(\mathbf{B}(z,t) + \sum_{n=1}^{\infty} b_{n,0} z^n\Big) \\ &= zt^2\,\mathbf{B}(z,t)\end{aligned}$$

Hence the right hand side sums to $z(1+t)^2\,\mathbf{B}(z,t) - z^2 t$, giving the identity

$$(1+z)\,\mathbf{I}(z,t) - zt = z(1+t)^2\,\mathbf{B}(z,t) \tag{21}$$

Using Eq. (19), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} z^n t^m \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\min(n,m)}\sum_{(m_1,\ldots,m_p;\,m-p+1)}\sum_{(n_1,\ldots,n_p;\,n)} (i_{n_1,m_1})\cdots(i_{n_p,m_p}) z^n t^m \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{p=1}^{\min(n,m)}\sum_{(m_1,\ldots,m_p;\,m-p+1)}\sum_{(n_1,\ldots,n_p;\,n)} (i_{n_1,m_1} z^{n_1} t^{m_1})\cdots(i_{n_p,m_p} z^{n_p} t^{m_p}) t^{p-1} \\ &= \sum_{p=1}^{\infty}\left(\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} i_{k,\ell} z^k t^\ell\right)^p t^{p-1},\end{aligned}$$

and hence

$$\mathbf{B}(z,t) = \frac{\mathbf{I}(z,t)}{1 - t\,\mathbf{I}(z,t)}. \tag{22}$$

Thus we obtained the identity defining $\mathbf{I}(z,t)$ as

$$(1+z)\,\mathbf{I}(z,t) - zt = z(1+t)^2\frac{\mathbf{I}(z,t)}{1 - t\,\mathbf{I}(z,t)} \tag{23}$$

Solving this quadratic equation in $\mathbf{I}(z,t)$ and using the initial conditions, we found

$$\mathbf{I}(z,t) = \frac{1-2tz-\sqrt{1-4zt-4zt^2}}{2t(z+1)}, \tag{24}$$

$$\mathbf{B}(z,t) = \frac{1-2zt-2zt^2-\sqrt{1-4zt-4zt^2}}{2t(1+t)^2 z}, \tag{25}$$

$$\mathbf{D}(z,t)=\frac{2z^2t^3+2z^2t^2-3zt^2-4zt+1+(zt^{!}+2zt-1)\sqrt{1-4zt-4zt^2}}{2t(1+t)^2z(1+z)}. \tag{26}$$

We can also obtain the bivariate generating series for $a_{n,m}$.

$$\begin{aligned}
\mathbf{A}(z,t) &= \sum_{n=0}^{\infty}\left(a_{n,0}z^n + a_{n,1}z^n t + \sum_{m=2}^{\infty} a_{n,m}z^n t^m\right) \\
&= t + \sum_{m=2}^{\infty} a_{0,m}t^m + \sum_{n=1}^{\infty}\left(\sum_{m=2}^{\infty} a_{n,m}z^n t^m\right) \\
&= t + \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}(2\,b_{n,m-1} + b_{n,m-2})z^n t^m \\
&= t + \sum_{n=1}^{\infty}\left(t\sum_{m=1}^{\infty} 2\,b_{n,m}z^n t^m\right) + t^2\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} b_{n,m}z^n t^m \\
&= t + 2t\,\mathbf{B}(z,t) + t^2\,\mathbf{B}(z,t).
\end{aligned}$$

The bivariate generating function $\mathbf{A}(z,t)$ is given by

$$\mathbf{A}(z,t) = \frac{2+t-2zt-2zt^2-(2+t)\sqrt{1-4zt-4zt^2}}{2(1+t)^2 z}. \tag{27}$$

Interesting sequences and counting information can be derived from these functions. For example, the power series expansion for $\mathbf{A}(z,t)$ in low degrees in z, with coefficient in t accurate up to $\mathcal{O}(t^{16})$, is

$$t + (2t^2+t^3)z + (4t^3+6t^4+2t^5)z^2 + (10t^4+25t^5+20t^6+5t^7)z^3 + \cdots.$$

From this, we can read off that there are 60 associate `RBWs` in R with three pairs of brackets, and 10 of these have arity 4, 25 have arity 5, 20 have arity 6, and 5 have arity 7. By expanding the series using t as the main variable, with coefficient in z accurate up to $\mathcal{O}(z^{10})$ we have

$$t + 2zt + (z+4z^2)t^3 + (6z^2+10z^3)t^4 + (2z^2+25z^3+28z^4)t^5 + \cdots.$$

From this we see that there are 55 associate `RBWs` in R with arity 5, and 2 of these have degree 2, 25 have degree 3 and 28 have degree 4. By specializing $z = 1$, we obtain a sequence for the number of associates in R with arity m, $(m = 1, 2, ...)$

$$1, 2, 5, 16, 55, 202, 773, 3052, \ldots$$

This sequence is new and not in the Sloane data base.

Theorem 3.3 *The generating series for* $r_{n,m}$ *is*

$$\mathbf{R}(z,t) = \frac{1 - \sqrt{1 - 4zt - 4zt^2}}{2tz}. \tag{28}$$

Proof: This follows by Rule (4) using $\mathbf{R}(z,t) = 1 + \mathbf{B}(z,t) + \mathbf{A}(z,t)$ and Eqs. (25) and (27). $\square$

Once again we have proved Theorem 3.1 (by putting $t = 1$ in Eqs. (24)–(28)). By specializing $z = 1$ to Eq. (28), we obtain the sequence A025227:

$$1, 2, 4, 12, 40, 144, 544, 2128, \ldots,$$

and thus give that sequence a new combinatorial interpretation. We easily obtain new sequences by specializing to other values, such as for $z = 2, 3, 4, 5, \ldots$:

$$\begin{aligned}
z = 2: &\quad 1, 3, 12, 66, 408, 2712, 18912, 136488, \ldots, \\
z = 3: &\quad 1, 4, 24, 192, 1728, 16704, 169344, \ldots, \\
z = 4: &\quad 1, 5, 40, 420, 4960, 62880, 835840, \ldots, , \\
z = 5: &\quad 1, 6, 60, 780, 1140, 178800, 2940000, \ldots, .
\end{aligned}$$

Moreover, this result is more refined than Theorem 3.1 and its corollary. Indeed, we note that Eq. (28) is related to the well-known generating series

$$\mathbf{C}(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z} \tag{29}$$

for the Catalan sequence C_n. We have clearly:

$$\begin{aligned}
\mathbf{R}(z,t) &= \frac{1-\sqrt{1-4zt(1+t)}}{2tz} = (1+t)\,\mathbf{C}(zt(1+t)) \\
&= \sum_{n=0}^{\infty} C_n z^n t^n (1+t)^{n+1} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n+1} \binom{n+1}{j} C_n z^n t^{n+j} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=n}^{2n+1} \binom{n+1}{m-n} C_n z^n t^m \right)
\end{aligned}$$

and hence

$$r_{n,m} = \begin{cases} \binom{n+1}{m-n} C_n & \text{if } n \leqslant m \leqslant 2n+1, n \geqslant 0; \\ \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

This result not only provides the proof anew that $R(n)$ has $2^{n+1}C_n$ `RBWs`, but also that these are distributed by their arities from n to $2n+1$ according to the binomial theorem. In a similar fashion, using Eq. (25) and a modified version of Eq. (29):

$$\overline{\mathbf{C}}(z) = \sum_{n=1}^{\infty} C_n z^n = \frac{1-2z-\sqrt{1-4z}}{2z}$$

the doubly-indexed sequence for the number of bracketed `RBWs` with degree n and arity m has the same property, that is,

$$b_{n,m} = \begin{cases} \binom{n-1}{m-n} C_n & \text{if } n \leqslant m \leqslant 2n-1, n \geqslant 1; \\ \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

This latter distribution, like the one for $R(n,m)$, was first observed by experimental computations, but the proof is not obvious because among the C_n ways to set up the structure of n pairs of balanced brackets, the number of ways to insert m x's to form bracketed `RBWs` (or `RBWs` in the case of $R(n,m)$) depends on the individual bracket structure (and sometimes, this number can be zero). An example that illustrates this observation is the set $B(3,4)$ which has the following 10 elements. The $C_3 = 5$ possible bracket structures (which

correspond to the 5 possible rooted trees with 4 vertices) are shown on the left.

structure	count	bracketed `RBWs`
$\lfloor\lfloor\lfloor\rfloor\rfloor\rfloor$	4	$\lfloor x\lfloor x\lfloor x\rfloor x\rfloor\rfloor, \lfloor\lfloor x\lfloor x\rfloor x\rfloor x\rfloor, \lfloor x\lfloor x\lfloor x\rfloor\rfloor x\rfloor, \lfloor x\lfloor\lfloor x\rfloor x\rfloor x\rfloor$
$\lfloor\lfloor\rfloor\lfloor\rfloor\rfloor$	2	$\lfloor x\lfloor x\rfloor x\lfloor x\rfloor\rfloor, \lfloor\lfloor x\rfloor x\lfloor x\rfloor x\rfloor$
$\lfloor\lfloor\rfloor\rfloor\lfloor\rfloor$	2	$\lfloor x\lfloor x\rfloor\rfloor x\lfloor x\rfloor, \lfloor\lfloor x\rfloor x\rfloor x\lfloor x\rfloor$
$\lfloor\rfloor\lfloor\lfloor\rfloor\rfloor$	2	$\lfloor x\rfloor x\lfloor x\lfloor x\rfloor\rfloor, \lfloor x\rfloor x\lfloor\lfloor x\rfloor x\rfloor$
$\lfloor\rfloor\lfloor\rfloor\lfloor\rfloor$	0	

3.3 Algorithm for generating R

It is also interesting to note that the two-variable generating functions studied above allow us to obtain a new and more effective way to generate, say $B(n,m)$ recursively, from constructions using bracketed `RBWs` alone. From Eq. (21) and Eq. (22), we obtain the identity satisfied by $\mathbf{B}(z,t)$:

$$\mathbf{B}(z,t) - zt = 2zt(1+t)\,\mathbf{B}(z,t) + zt(1+t)^2\,\mathbf{B}(z,t)^2. \tag{32}$$

Substituting the definition of $\mathbf{B}(z,t)$ into the above yields:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} z^n t^m - zt \\ &= 2zt(1+t)\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} z^n t^m\right) + zt(1+t)^2\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{n,m} z^n t^m\right)^2\end{aligned}$$

from which we obtain, for $(n,m) \neq (1,1)$,

$$\begin{aligned} b_{n,m} \;=\;& 2\,b_{n-1,m-1} + 2\,b_{n-1,m-2} + \sum_{\substack{(n_1,n_2;\,n-1)\\(m_1,m_2;\,m-1)}} b_{n_1,m_1} b_{n_2,m_2} \\ &+ 2\sum_{\substack{(n_1,n_2;\,n-1)\\(m_1,m_2;\,m-2)}} b_{n_1,m_1} b_{n_2,m_2} + \sum_{\substack{(n_1,n_2;\,n-1)\\(m_1,m_2;\,m-3)}} b_{n_1,m_1} b_{n_2,m_2} \end{aligned} \tag{33}$$

or, in explicit summation form:

$$
\begin{aligned}
b_{n,m} \quad = \quad & 2\, b_{n-1,m-1} + 2\, b_{n-1,m-2} + \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-2} b_{k,\ell} b_{n-1-k,m-1-\ell} \\
& + 2 \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-3} b_{k,\ell} b_{n-1-k,m-2-\ell} \; + \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-4} b_{k,\ell} b_{n-1-k,m-3-\ell} \,.
\end{aligned} \tag{34}
$$

This recursion suggests that we can generate $B(n,m)$ (and hence also $R(n,m)$) efficiently and irredundantly from sets $B(k,\ell)$ with $k < n, \ell < m$ in some manner. This is indeed the case.

Theorem 3.4 *Given positive integers n and m, the algorithm below returns the sets $B(n,m)$ (resp. $I(n,m)$, resp. $D(n,m)$) of bracketed (resp. indecomposable, resp. decomposable)* `RBWs` *of degree n and arity m. For $n \geqslant 4$, $B(n,m)$ is expressed as the disjoint union of sets constructed from $B(k,\ell)$ with $k < n$ and $\ell < m$ according to* Eq. (34).

Algorithm for bracketed `RBWs` of degree n and arity m

Input: Positive integers n, m

Output:

(a) the set $B(n,m)$ of bracketed `RBWs` of degree n and arity m,

(b) the set $I(n,m)$ of indecomposable `RBWs` of degree n and arity m,

(c) the set $D(n,m)$ of decomposable `RBWs` of degree n and arity m.

Step 0. If not$(n \leqslant m \leqslant 2n-1)$, then return three empty sets.
Generate all bracketed (resp. indecomposable, resp. decomposable) `RBWs` with degree $k \leqslant 3$ and arity ℓ between k and $2k-1$.
If $n \leqslant 3$ then return $B(n,m)$, $I(n,m)$, $D(n,m)$.

Step 1. For each `RBW` $w \in B(n-1,m-1)$, form two `RBWs`
$f_{1,1}(w) = \lfloor x\, w \rfloor$ and $f_{1,2}(w) = \lfloor w\, x \rfloor$.

Step 2. For each `RBW` $u \in B(n-1,m-2)$, form the `RBWs` $f_2(u) = \lfloor x\, u\, x \rfloor$

Step 3. For each $k = 1 \ldots (n-2)$, each $\ell = 1 \ldots m-2$,
and each pair of `RBWs` $(v,y) \in I(k,\ell) \times B(n-1-k, m-1-\ell)$,
form the `RBW` $f_3(v,y) = \lfloor v\, x\, y \rfloor$

Step 4. For each $k = 1 \ldots (n-2)$, each $\ell = 1 \ldots m-2$,
and each pair of `RBWs` $(v,y) \in D(k,\ell) \times B(n-1-k, m-1-\ell)$
form the `RBW` $f_4(v,y) = \lfloor v \rfloor\, x\, y$

Step 5. For each `RBW` $u \in B(n-1,m-2)$, form the `RBWs` $f_5(u) = \lfloor x \rfloor\, x\, u$.

Step 6. For each $k = 1 \ldots (n-2)$, each $\ell = 1 \ldots m-3$,
and each pair of `RBWs` $(v, y) \in B(k, \ell) \times B(n-1-k, m-2-\ell)$,
form the two `RBWs` $f_{6,1}(v, y) = \lfloor x\,v \rfloor\, x\,y$ and $f_{6,2}(v, y) = \lfloor v\,x \rfloor\, x\,y$.

Step 7. For each $k = 1 \ldots (n-2)$, each $\ell = 1 \ldots m-4$,
and each pair of `RBWs` $(v, y) \in B(k, \ell) \times B(n-1-k, m-3-\ell)$,
form the `RBW` $f_7(v, y) = \lfloor x\,v\,x \rfloor\, x\,y$.

Step 8. Return the union of all the `RBWs` formed in Steps 1–3 as $I(n, m)$,
the union of all the `RBWs` formed in Steps 4–7 as $D(n, m)$,
and the union of $I(n, m)$ and $D(n, m)$ as $B(n, m)$.

Proof: By Eq. (32), we know that $B(n, m)$, $I(n, m)$, and $D(n, m)$ are empty if $m < n$ or $m > 2n-1$. The cases when $n \leqslant 3$ is taken care of in Step 0. So suppose $n \geqslant 4$ and $n \leqslant m \leqslant 2n-1$. We first note the disjoint union $B(n, m) = I(n, m) \cup D(n, m)$ from Eq. (6).

Any word z in $I(n, m)$ is of the form $\lfloor z' \rfloor$ where $z' \in R(n-1, m)$ by Eq. (7). Then one and exactly one of the following statements on z' is true.

Case–1. Either z' starts with x but does not end with x, implying $z = \lfloor x\,w \rfloor$ with $w \in B(n-1, m-1)$; or
z' does not start with x but ends with x, implying $z = \lfloor w\,x \rfloor$ with $w \in B(n-1, m-1)$;

Case–2. z' starts with x and ends with x, implying $z = \lfloor x\,u\,x \rfloor$ with $u \in B(n-1, m-2)$;

Case–3. z' neither starts nor ends with x. Then z' must be decomposable and so $z' \in D(n-1, m)$. Let v be the leftmost indecomposable subword of z' (see Eq. (8)) and let k be the degree of v and ℓ be the arity of v. Then $1 \leqslant k \leqslant n-2$, $1 \leqslant \ell \leqslant m-2$, and $v \in I(k, \ell)$. Moreover, we can write z' uniquely as $v\,x\,y$ and $z = \lfloor v\,x\,y \rfloor$ where $y \in B(n-1-k, m-1-\ell)$.

This proves that $I(n, m)$ is a subset of the set of `RBWs` generated by Steps 1–3. Conversely, any `RBW` z generated by Steps 1–3 clearly belongs to $I(n, m)$ by definition. Thus $I(n, m)$ is precisely the set of `RBWs` generated by Steps 1–3, and moreover, the sets generated in each of these steps are disjoint. This shows that

$$i_{n,m} = 2\, b_{n-1,m-1} + b_{n-1,m-2} + \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-2} i_{k,\ell}\, b_{n-1-k,m-1-\ell}. \tag{35}$$

Now consider $z \in D(n,m)$. As in Case–3 above, z is of the form $\lfloor z' \rfloor\, x\, y$ for a unique `RBW` z' ($\lfloor z' \rfloor$ being the leftmost indecomposable subword of z) and a unique bracketed `RBW` y. Let k be the degree of z'. Then $1 \leqslant k \leqslant n-2$, and one and exactly one of the following statements on z' is true.

Case–4. z' neither starts nor ends with an x, implying that z' is decomposable. Denoting z' by v, and letting ℓ be the arity of v, we have $z = \lfloor v \rfloor\, x\, y$ with

$$(v, y) \in D(k, \ell) \times B(n-1-k, m-1-\ell),$$

and $1 \leqslant \ell \leqslant m-2$.

Case–5. z' is x, implying $z = \lfloor\, x \,\rfloor\, x\, y$ and $y \in B(n-1, m-2)$.

Case–6. Either z' starts with x but does not end with x, implying $z = \lfloor x\, v \,\rfloor\, x\, y$ for some $(v,y) \in B(k,\ell) \times B(n-1-k, m-2-\ell)$ where $1 \leqslant \ell \leqslant m-3$; or
z' does not start with x but ends with x, implying $z = \lfloor v\, x \rfloor\, x\, y$ with $(v,y) \in B(k,\ell) \times B(n-1-k, m-2-\ell)$ where $1 \leqslant \ell \leqslant m-3$;

Case–7. z' starts with x and ends with x (but is not x), implying $z = \lfloor x\, v\, x \rfloor\, x\, y$ with $(v,y) \in B(k,\ell) \times B(n-1-k, m-3-\ell)$ where $1 \leqslant \ell \leqslant m-4$;

Note that Cases 4–7 correspond respectively to Steps 4–7 of the algorithm and generate disjoint subsets of $D(n,m)$. This shows that $D(n,m)$ is precisely the set of `RBWs` generated by these steps and hence

$$\begin{aligned} d_{n,m} \quad &= \quad \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-2} d_{k,\ell}\, b_{n-1-k,m-1-\ell} + b_{n-1,m-2} \\ &\quad + 2 \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-3} b_{k,\ell}\, b_{n-1-k,m-2-\ell} \; + \sum_{k=1}^{n-2} \sum_{\ell=1}^{m-4} b_{k,\ell}\, b_{n-1-k,m-3-\ell}\,. \end{aligned} \tag{36}$$

Combining equations (35) and (36) finishes the proof (and provides a second, constructive, proof of Eq. (34)). $\square$

4 One generator and one operator: arbitrary exponent case

We now consider the more general cases. In this section, we generalize previous results to the cases of one generator x and one operator $P = \lfloor\ \rfloor$ without requiring these to be idempotent. Referring to Example 2.6, we have again $q = 1$, but now $v = v_1$ is arbitrary (including $v_1 = \infty$). Moreover, we also restrict the set of Rota-Baxter words $\mathfrak{M}^1(1, (v))$ to those where the number of consecutive applications of the operator P is bounded by a given u (which may also be ∞, in which case there will be no restriction at all).

4.1 Notations

For this section, we now introduce new terminology and notations. For any `RBW` w, and operator $P = \lfloor\ \rfloor$ occurring in w, a P-**run** is any occurrence in w of consecutive compositions of $\lfloor\ \rfloor$ of maximal length (that is, of immediately nested $\lfloor\ \rfloor$, where the length is the number of consecutive applications of P). Recall from Definition 2.5 for any generator x, an x-**run** is any occurrence in w of consecutive (algebraic) products of x of maximal length. We denote a P-run by $P^{(\mu)}$ or $\lfloor\ \rfloor^{(\mu)}$ if μ is its run length, and an x-run by x^{ν} if ν is its run length. When μ or ν is 1, we shall often omit the superscript. Let u, v be either positive integers or ∞ and let $R_{u,v}$ be the subset of `RBWs` (including $\emptyset$) where the length of P-runs is $\leqslant u$ and the length of x-runs is $\leqslant v$. These subsets are potential canonical bases of Rota-Baxter algebras on one generator. We have seen in Section 2 that $R_{\infty,v}$ is the canonical basis of the free Rota-Baxter algebra $Ш^{NC,0}(1,(v))$ (Example 2.6). Also $R_{1,1}$ is the canonoical basis of the free Rota-Baxter algebra with one idempotent generator and one idempotent operator considered in Section 3 (see also Aguiar and Moreira (2005)).

For convenience, we say the operator $P = \lfloor\ \rfloor$ has **exponent** u and the generator x has **exponent** v if we are enumerating the set $R_{u,v}$. This would be the case for Rota-Baxter algebras where the generator x satisfies $x^{v+1} = x$ and the operator P satisfies $P^{(u+1)}(w) = P(w)$ for any w. In this section, our enumeration on `RBWs` is valid for any unary operator P. It is not clear under what conditions a Rota-Baxter operator P would have exponent u for $u \geqslant 2$.

For $n \geqslant 1$, let $R_{u,v}(n)$ be the subset of $R_{u,v}$ consisting of all `RBWs` of degree n, and for $m \geqslant 1$, let $R_{u,v}(n, m)$ be the subset of $R_{u,v}$ consisting of `RBWs` with

degree n and arity m. Moreover, for $1 \leqslant k \leqslant n$, we let $R_{u,v}(n,m;k)$ be the subset of $R_{u,v}(n,m)$ consisting of `RBWs` where the n pairs of balanced brackets are distributed into exactly k P-runs, and for $1 \leqslant \ell \leqslant m$, we let $R_{u,v}(n,m;k,\ell)$ be the subset of $R_{u,v}(n,m)$ consisting of `RBWs` where the n pairs of balanced brackets are distributed into exactly k P-runs, and the m x's are distributed into exactly ℓ x-runs. Except for $R_{u,v}(n)$, these subsets are all finite sets, even when u, v are infinite, and we shall denote their corresponding cardinalities by replacing R by the lower case r. Thus, for example, $r_{u,v}(n,m;k,\ell)$ is the cardinality for $R_{u,v}(n,m;k,\ell)$ and the count $r_{n,m}$ of Section 3 is now denoted by $r_{1,1}(n,m)$. This convention will be used for all other (finite) sets of `RBWs` we may introduce later.

As an example for the above terms and notations, the `RBW`

$$w = x^2\lfloor x\lfloor x^3\rfloor^{(2)}x^2\rfloor = xx\lfloor x\lfloor\lfloor xxx\rfloor\rfloor xx\rfloor$$

is an element in $R_{u,v}(3,8;2,4)$ for any $u \geqslant 2, v \geqslant 3$ since the 3 pairs of balanced brackets occur in 2 P-runs of run-lengths 1 and 2, and the 8 x's occur in 4 x-runs of run-lengths 2, 1, 3, 2.

We also define the following generating series:

$$\mathbf{R}_{u,v}(z) = \sum_{n=0}^{\infty} r_{u,v}(n)z^n \tag{37}$$

$$\mathbf{R}_{u,v}(z,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{u,v}(n,m)z^n t^m \tag{38}$$

$$\mathbf{R}_{u,v}(z,t;\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} r_{u,v}(n,m;k)z^n t^m \zeta^k \tag{39}$$

$$\mathbf{R}_{u,v}(z,t;\zeta,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} r_{u,v}(n,m;k,\ell)z^n t^m \zeta^k \theta^\ell \tag{40}$$

We divide our study into three parts. In Section 4.2, we recall some results on compositions. In Section 4.3, we develop enumeration formulae relating the general cases to the idempotent case. In the remaining Section 4.4, we compute the generating functions, and sketch an algorithm for generating the sets $R_{u,v}(n,m)$.

We recall (see, for example, MacMahon (1984)) a well-known result on compositions (also called ordered partitions) of a positive integer m. Let $G(m, \ell, v)$ be the set of compositions of the integer m into ℓ positive integer parts, with each part at most v and let $g(m, \ell, v)$ be the size of this set. When v is finite, it is easy to see from the multinomial expansion of

$$(t + t^2 + \cdots + t^v)^\ell$$

by collecting the coefficients of t^m that

$$g(m, \ell, v) = \sum_{\substack{\ell_1+\cdots+\ell_v=\ell \\ \ell_1+2\ell_2+\cdots+v\ell_v=m}} \binom{\ell}{\ell_1, \cdots, \ell_v}$$

where $\ell_j \geqslant 0$ is the number of times t^j is chosen among the ℓ factors of the product. It also follows that

$$\sum_{m=1}^{\infty} g(m, \ell, v) t^m = (t + t^2 + \cdots + t^v)^\ell.$$

Thus, for given ℓ and v, we find that the left-hand-side is the generating function $\mathbf{G}_{\ell,v}(t)$ for g with respect to m, namely:

$$\mathbf{G}_{\ell,v}(t) := t^\ell \left(\frac{1 - t^v}{1 - t} \right)^\ell . \tag{41}$$

In a similar but simpler way, by expanding $(t + t^2 + \cdots)^\ell$ when $v = \infty$, we see that

$$g(m, \ell, \infty) = \binom{m-1}{\ell-1}, \tag{42}$$

which is the number of compositions of m into ℓ parts, with no restrictions on the size of each part. We have an associated generating function

$$\mathbf{G}_{\ell,\infty}(t) := \sum_{m=1}^{\infty} g(m, \ell, \infty) t^m = \sum_{m=1}^{\infty} \binom{m-1}{\ell-1} t^m = \left(\frac{t}{1-t} \right)^\ell . \tag{43}$$

At the other extreme case, when $v = 1$, then $g(m, \ell, 1) = \delta_{m,\ell}$ (Kronecker's δ) with $\mathbf{G}_{\ell,1}(t) = t^\ell$.

Regarding the power series ring $\mathbb{Z}[[t]]$ over the ring $\mathbb{Z}$ of integers as the completion of the polynomial ring $\mathbb{Z}[t]$ in the usual sense, we have

$$\mathbf{G}_{\ell,\infty}(t) = \lim_{v\to\infty} \mathbf{G}_{\ell,v}(t).$$

This implies that

$$g(m,\ell,\infty) = \lim_{v\to\infty} g(m,\ell,v).$$

In fact, $g(m,\ell,\infty) = g(m,\ell,v)$ for all $v \geqslant m$.

Lemma 4.1 *For any power series* $\mathbf{F}(t) = \sum_{\ell\geqslant 1} a_\ell t^\ell$ *and* $1 \leqslant v \leqslant \infty$*, we have*

$$\mathbf{F}\circ\mathbf{G}_{1,v}(t) = \sum_{m\geqslant 1}\Big(\sum_{\ell=1}^{m} g(m,\ell,v)a_\ell\Big)t^m.$$

Proof: We have

$$\begin{aligned}
\mathbf{F}\circ\mathbf{G}_{1,v}(t) &= \sum_{\ell\geqslant 1} a_\ell \left(\sum_{j=1}^{v} t^j\right)^{\ell} \\
&= \sum_{\ell\geqslant 1} a_\ell \mathbf{G}_{\ell,v}(t) \\
&= \sum_{\ell\geqslant 1}\sum_{m=1}^{\infty} a_\ell g(m,\ell,v)t^m \\
&= \sum_{m\geqslant 1}\Big(\sum_{\ell=1}^{m} g(m,\ell,v)a_\ell\Big)t^m,
\end{aligned}$$

since $g(m,\ell,v) = 0$ if $\ell > m$. $\square$

4.3 Enumeration

Recall that in the case when the Rota-Baxter operator $\lfloor\ \rfloor$ and the generator x are both idempotent (that is, each of exponent 1), we had studied in Section 3 the set $R_{1,1}(n) = R(n)$, which is the set of all `RBWs` of degree n and the set $R_{1,1}(n,m) = R(n,m)$, which is the set of all `RBWs` of degree n and arity m. It was shown in Eq. (28) that the doubly indexed sequence $r_{n,m} = r_{1,1}(n,m)$ has the generating function:

$$\mathbf{R}_{1,1}(z,t) := \mathbf{R}(z,t) = \sum_{n,m\geqslant 0} r_{1,1}(n,m)z^n t^m = \frac{1-\sqrt{1-4zt-4zt^2}}{2tz}.$$

We have the disjoint union:

$$R_{u,v}(n,m) = \coprod_{k=1}^{n} \coprod_{\ell=1}^{m} R_{u,v}(n,m;k,\ell). \tag{44}$$

We note that in order for $R_{1,1}(n,m;k,\ell)$ to be non-empty, we must have $n = k$ and $m = \ell$, in which case, we have $R_{1,1}(n,m;k,\ell) = R_{1,1}(k,\ell)$. For any $w \in R_{1,1}(k,\ell)$, w has exactly k pairs of balanced brackets, no two of which are immediately nested, and w has exactly ℓ x's no two of which are next to each other. We define below what might be called a "collapsing map":

$$\Psi_{u,v,n,m} : R_{u,v}(n,m;k,\ell) \longrightarrow R_{1,1}(k,\ell). \tag{45}$$

Given $w \in R_{u,v}(n,m;k,\ell)$, we define $\Psi_{u,v,n,m}(w)$ to be the `RBW` $w_1 \in R_{1,1}(k,\ell)$ obtained by replacing each of the k P-runs that appears in w by a single P and each of the x-runs appearing in w by a single x. This map is clearly surjective for each pair (k,ℓ). We shall refer to w_1 as the **collapse** of w. The map $\Psi_{u,v,n,m}$ technically depends on k,ℓ, but for simplicity, we will omit that in the notation. Moreover by Eq. (44), we may by abuse of language, extend all these maps uniquely to a single one:

$$\Psi_{u,v,n,m} : R_{u,v}(n,m) \longrightarrow \coprod_{k=1}^{n} \coprod_{\ell=1}^{m} R_{1,1}(k,\ell).$$

For each `RBW` $w \in R_{u,v}(n,m;k,\ell)$, w has exactly k P-runs and exactly ℓ x-runs. For $1 \leqslant i \leqslant k$, let n_i be the run-length of the i-th P-run of w and for $1 \leqslant j \leqslant \ell$, let m_j be the run-length of the j-th x-run of w. We have $1 \leqslant n_i \leqslant u$ (since P has exponent u) and $1 \leqslant m_j \leqslant v$ (since x has exponent v). Then to each w there corresponds the triplet $(w_1, \vec{n}, \vec{m})$ where $w_1 = \Psi_{u,v,n,m}(w) \in R_{1,1}(k,\ell)$, a composition $\vec{n} = (n_1, \ldots, n_k)$ of n into k parts with each part n_i no bigger than u, and a composition $\vec{m} = (m_1, \ldots, m_\ell)$ of m into ℓ parts with no part m_j bigger than v. Conversely, every such triplet determines a unique $w \in R_{u,v}(n,m;k,\ell)$ such that $\Psi_{u,v,n,m}(w) = w_1$. Using the sets $G(m,\ell,v)$ for compositions defined in Section 4.2, we have a bijection:

$$R_{u,v}(n,m;k,\ell) \longleftrightarrow R_{1,1}(k,\ell) \times G(n,k,u) \times G(m,\ell,v) \tag{46}$$

Theorem 4.2 *With the notations established in* Sections 4.1*,* 4.2 *and above, the numbers $r_{u,v}(n,m)$ of* `RBWs` *of degree n and arity m in the set $R_{u,v}$ of* `RBWs` *with one generator x of exponent v and one operator P of exponent u are given by:*

(1) *For* $1 \leqslant u \leqslant \infty$, $r_{u,1}(n,m) = \sum_{k=1}^{n} g(n,k,u) r_{1,1}(k,m)$;

(2) *For* $1 \leqslant v \leqslant \infty$, $r_{1,v}(n,m) = \sum_{\ell=1}^{m} g(m,\ell,v) r_{1,1}(n,\ell)$;

(3) *For* $1 \leqslant u, v \leqslant \infty$, $r_{u,v}(n,m)$ *is given by any of the three expressions:*

 (a) $\sum_{\ell=1}^{m} g(m,\ell,v) r_{u,1}(n,\ell)$,

 (b) $\sum_{k=1}^{n} g(n,k,u) r_{1,v}(k,m)$,

 (c) $\sum_{k=1}^{n} \sum_{\ell=1}^{m} g(n,k,u) g(m,\ell,v) r_{1,1}(k,\ell)$.

Proof: The proofs are all similar. For example, 3(c) follows from the bijection in Eq. (46) and the disjoint union in Eq. (44). □

Corollary 4.3 *Suppose one or both of* u *and* v *are* ∞. *Then we have*

$$r_{\infty,1}(n,m) = \sum_{k=1}^{n} \binom{n-1}{k-1} \binom{k+1}{m-k} C_k,$$

$$r_{1,\infty}(n,m) = \sum_{\ell=1}^{m} \binom{m-1}{\ell-1} \binom{\ell+1}{n-\ell} C_k$$

$$r_{\infty,\infty}(n,m) = \sum_{k=1}^{n} \sum_{\ell=1}^{m} \binom{n-1}{k-1} \binom{m-1}{\ell-1} \binom{k+1}{\ell-k} C_k.$$

4.4 Generating Functions and Enumeration Algorithm

We now find the generating functions $\mathbf{R}_{u,v}(z,t)$ of the number sequences $r_{u,v}(n,m)$ for $1 \leqslant u, v \leqslant \infty$ (see Eqs. (38)–(40) for definitions and notations). Recall from Eqs. (28)–(29), we have

$$\mathbf{R}_{1,1}(z,t) = \mathbf{R}(z,t) = \frac{1 - \sqrt{1 - 4zt(1+t)}}{2zt} = (1+t)\mathbf{C}(zt(1+t))$$

where

$$\mathbf{C}(z) = \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1-4z}}{2z}$$

is the generating function of the Catalan numbers.

Theorem 4.4 *Let $1 \leqslant u, v \leqslant \infty$. The generating function $\mathbf{R}_{u,v}(z,t)$ for the number $r_{u,v}(n,m)$ of* `RBWs` *in a Rota-Baxter algebra with one operator P with exponent u and one generator x with exponent v is given by:*

(1) $\mathbf{R}_{u,1}(z,t) = \mathbf{R}_{1,1}\left(\mathbf{G}_{1,u}(z),\ t\right)$.
(2) $\mathbf{R}_{1,v}(z,t) = \mathbf{R}_{1,1}\left(z, \mathbf{G}_{1,v}(t)\right)$.
(3) $\mathbf{R}_{u,v}(z,t) = \mathbf{R}_{1,1}\left(\mathbf{G}_{1,u}(z), \mathbf{G}_{1,v}(t)\right)$.

where $\mathbf{G}_{1,u}$ and $\mathbf{G}_{1,v}$ are given by Eq. (41) (*for finite u, v*) *and by* Eq. (43) (*for infinite u, v*).

Proof: The proofs follow from Theorem 4.2 and Lemma 4.1. We just prove case (2):

$$\begin{aligned}
\mathbf{R}_{1,v}(z,t) &= \sum_{n,m\geqslant 1} r_{1,v}(n,m) z^n t^m \\
&= \sum_{n,m\geqslant 1} \sum_{\ell=1}^{m} g(m,\ell,v) r_{1,1}(n,\ell) z^n t^m \\
&= \sum_{m\geqslant 1} \left(\sum_{\ell=1}^{m} g(m,\ell,v) \left(\sum_{n\geqslant 1} r_{1,1}(n,\ell) z^n \right) \right) t^m \\
&= \sum_{\ell\geqslant 1} \left(\sum_{n\geqslant 1} r_{1,1}(n,\ell) z^n \right) \left(\mathbf{G}_{1,v}(t)\right)^{\ell} \\
&= \mathbf{R}_{1,1}\left(z,\ \mathbf{G}_{1,v}(t)\right). \ \square
\end{aligned}$$

Corollary 4.5 *Suppose one or both of u, v are ∞, then the generating function for $r_{u,v}(n,m)$ is given by*

$$\begin{aligned}
\mathbf{R}_{\infty,1}(z,t) &= \mathbf{R}_{1,1}\left(\frac{z}{1-z}, t\right), \\
\mathbf{R}_{1,\infty}(z,t) &= \mathbf{R}_{1,1}\left(z, \frac{t}{1-t}\right), \\
\mathbf{R}_{\infty,\infty}(z,t) &= \mathbf{R}_{1,1}\left(\frac{z}{1-z}, \frac{t}{1-t}\right).
\end{aligned}$$

We end this section with a brief description for an algorithm to enumerate the sets $R_{u,v}(n,m)$. The details will be left out since by means of the disjoint union in Eq. (44) and the bijection in Eq. (46), this is fairly straight forward. We already have an algorithm (see Theorem 3.4) for the enumeration of $R_{1,1}(k,\ell)$ for any positive k, ℓ. We need an algorithm to generate all the compositions of m in $G(m,\ell,v)$, which would of course generate $G(n,k,u)$,

too. Now the set of compositions $\vec{m} = (m_1, \ldots, m_\ell)$ of m into exactly ℓ parts without restrictions on the parts can be enumerated by readily available, efficient, and well-known algorithms (see COMP_NEXT of SUBSET library in Nijenhuis and Wilf (1978) for example). Those compositions whose parts violate the restrictions $m_j \leqslant v$ can be easily discarded by modifying the code.

5 The general case: multiple generators and operators

In this section, we consider sets of `RBWs` with p unary operators $P_1, \ldots, P_p$ and q generators $x_1, \ldots, x_q$. Since our purpose here is enumeration, we will for now consider formal bracketed words with brackets $\lfloor_i\, \rfloor_i$ corresponding to P_i $(1 \leqslant i \leqslant p)$ and ignore any other properties of the operators.

While it is possible to give a definition generalizing that of Definition 2.1, it is more straight-forward to generalize the grammar of Section 3. Since we do not need such details, we leave the specification of the grammar to the reader.

We adopt the convention that a vector quantity using the same symbol as the corresponding scalar quantity will have components with the same symbol but subscripted. For example, $\vec{P} = (P_1, \ldots, P_p)$ and $\vec{x} = (x_1, \ldots, x_q)$. The $\vec{P}$-exponent vector $\vec{u} = (u_1, \ldots, u_p)$ will mean that the operator P_i has exponent u_i, $1 \leqslant i \leqslant p$, and this means that we only consider `RBWs` in which the number of consecutive applications of the operator P_i is bounded by u_i for each i. We define P_i-runs (resp. x_j-runs) similarly to P-runs (resp. x-runs), treating each P_i (resp. x_j) as single operator (resp. generator). We shall call a run of P's (with whatever subscripts) a $\vec{P}$-run, and similarly define an $\vec{x}$-run. Sets of `RBWs` are defined also with the parameters vectorized. For a numerical vector such as $\vec{n}$, we let $|\vec{n}|$ denote its **norm**, which is the sum of its components. Thus, without further explanation, $|\vec{m}|$ will be the total $\vec{x}$-arity of a `RBW` w whose $\vec{x}$-arity vector is $\vec{m}$. As an example, when $p = q = 2$, the `RBW`

$$\begin{aligned} w &= x_1^3 x_2^4 P_1 P_2^{(3)}(x_1 x_2 P_1(x_1)) \\ &= x_1x_1x_1x_2x_2x_2x_2\lfloor_1\lfloor_2\lfloor_2\lfloor_2 x_1x_2\lfloor_1 x_1\rfloor_1\rfloor_2\rfloor_2\rfloor_2\rfloor_1 \end{aligned} \tag{47}$$

has three x_1-runs of lengths 3, 1, and 1; two x_2-runs of lengths 4 and 1; three $\vec{x}$-runs of lengths 7, 2, and 1; two P_1-runs of lengths 1 and 1; one P_2-run of

length 3, and two $\vec{P}$-runs of lengths 4 and 1. The $\vec{P}$-degree vector of w is $(2,3)$, its $\vec{P}$-degree is 5, its $\vec{x}$-degree vector is $(5,5)$, and its $\vec{x}$-arity is 10.

For any given positive integers p, q, and corresponding $\vec{P}$-exponent vector $\vec{u}$, $\vec{x}$-exponent vector $\vec{v}$, let $R_{\vec{u},\vec{v}}$ denote the set of `RBWs` with p operators and q generators with corresponding exponents vectors $\vec{u}$ and $\vec{v}$ respectively. The values p, q are implicitly given by the dimensions of the vectors $\vec{u}$ and $\vec{v}$ respectively. In particular $R_{\infty,\infty}$ is the set of all `RBWs` in the setting with one generator (of exponent ∞) and one operator (also of exponent ∞). Thus $R_{\infty,\infty}$ is the canonical basis for $Ш^{\mathrm{NC},0}(\mathbf{k}[x])$ (see Example 2.2). Without further explanations, the notations established in Section 4.1 will be generalized to their vectorized versions in the obvious way. For example, $R_{\vec{u},\vec{v}}(n,m;k,\ell)$ will denote the set of all `RBWs` of $R_{\vec{u},\vec{v}}$ with $\vec{P}$-degree n distributed into exactly k $\vec{P}$-runs, and $\vec{x}$-arity m distributed into exactly ℓ $\vec{x}$-runs. Furthermore, the cardinality of this set is denoted by $r_{\vec{u},\vec{v}}(n,m;k,\ell)$ and analogous to Eq. (44), we have the disjoint union:

$$R_{\vec{u},\vec{v}}(n,m) = \coprod_{k=1}^{n} \coprod_{\ell=1}^{m} R_{\vec{u},\vec{v}}(n,m;k,\ell). \tag{48}$$

5.1 The forgetful maps and coloring

We define another family of maps to relate the general case to the single operator and single generator case we have solved. Basically, the forgetful maps removes all the subscripts on the operators and generators. By composing the forgetful maps with the collapsing maps, we further relate the general case with the case when the single operator and single generator have exponent 1. These maps capture the structure of runs of `RBWs` in the general case. We now set up the notations to make these precise.

We define the "forgetful" maps:

$$\Phi_{\vec{u},\vec{v},n,m} : R_{\vec{u},\vec{v}}(n,m;k,\ell) \longrightarrow R_{\infty,\infty}(n,m;k,\ell) \tag{49}$$

by defining $\Phi_{\vec{u},\vec{v},n,m}(w)$, for each $w \in R_{\vec{u},\vec{v}}(n,m;k,\ell)$, to be the `RBW` w_∞ in $R_{\infty,\infty}$ when every $P_i = \lfloor_i \rfloor_i$, $1 \leqslant i \leqslant p$ that appears in w is replaced by $P = \lfloor\ \rfloor$ and every $x_j, 1 \leqslant j \leqslant q$ that appears in w is replaced by x. For example, for the word w in Eq. (47), we have

$$\Phi_{\vec{u},\vec{v},n,m}(w) = x^7 P^{(4)}(x^2 P(x)) = x^7 \lfloor x^2 \lfloor x \rfloor \rfloor^{(4)}.$$

The map $\Phi_{\vec{u},\vec{v},n,m}$ is clearly surjective and depends on $p, q, \vec{u}, \vec{v}, n, m, k, \ell$, but for notational brevity, we will not explicitly mention p, q, k, and ℓ. Again, by abuse of language and the disjoint unions in Eq. (48) and Eq. (44), we may extend $\Phi_{\vec{u},\vec{v},n,m}$ uniquely to a surjection:

$$\Phi_{\vec{u},\vec{v},n,m} : R_{\vec{u},\vec{v}}(n, m) \longrightarrow R_{\infty,\infty}(n, m).$$

We may further extend $\Phi_{\vec{u},\vec{v},n,m}$ to a surjection:

$$\Phi_{\vec{u},\vec{v}} : R_{\vec{u},\vec{v}} \longrightarrow R_{\infty,\infty}.$$

It is convenient to refer to the `RBW` $w_\infty = \Phi_{\vec{u},\vec{v}}(w)$ as the **monochrome image** of w and say w is a **coloring** of w_∞. A special case of particular importance is the following.

Consider an $\vec{x}$-run w which we may write in the form $x_{j_1}^{b_1} \cdots x_{j_\beta}^{b_\beta}$ where β is some positive integer and for $1 \leqslant d \leqslant \beta$, we have $1 \leqslant j_d \leqslant q$, $1 \leqslant b_d \leqslant v_{j_d}$, and if $d < \beta$, then $j_d \neq j_{d+1}$. The monochrome image of w will be x^b, where $b = b_1 + \cdots + b_\beta$. Conversely, for any positive integer b, any coloring of x^b is an $\vec{x}$-run w of this form. Thus there is a bijection $\sigma = \sigma_{b,q,\vec{v}}$ between the set of $\vec{x}$-runs w whose monochrome image is x^b and the set $C(b, q, \vec{v})$ of colorings of b identical objects in a row using up to q colors, say colors $1, 2, \ldots, q$, repetitions allowed, so that for $1 \leqslant j \leqslant q$, any run of color j has length no longer than v_j. Explicitly, given w, we define a coloring $\sigma(w) \in C(b, q, \vec{v})$ of the b objects using color j_1 for the first b_1 objects, then color j_2 for the next b_2 objects, etc. The number of such colorings is denoted by $c(b, q, \vec{v})$.

Similarly, for any positive integer a, we have a bijection $\pi = \pi_{a,p,\vec{u}}$ between the set of $\vec{P}$-runs whose monochrome image is $P^{(a)}$ and the set of colorings $C(a, p, \vec{u})$. We can also apply these coloring maps to $\vec{x}$-runs and $\vec{P}$-runs occurring in any `RBW` $w \in R_{\vec{u},\vec{v}}(n, m; k, \ell)$ whose monochrome image is a fixed $w_\infty = \Phi_{\vec{u},\vec{v},n,m}(w)$. More explicitly, given $w_\infty \in R_{\infty,\infty}(n, m; k, \ell)$, let $(w_1, \vec{n}, \vec{m})$ be the triplet corresponding to w_∞ according to the bijection from Eq. (46) for the case $u = v = \infty$, so that $\vec{n} = (n_1, \ldots, n_k)$ is a composition of n into k parts, $\vec{m} = (m_1, \ldots, m_\ell)$ is a composition of m into ℓ parts, and, in particular, w_∞ collapses to $w_1 \in R_{1,1}(k, \ell)$ (that is, $\Psi_{\infty,\infty,n,m}(w_\infty) = w_1$). This implies that for all $w \in (\Phi_{\vec{u},\vec{v}})^{-1}(w_\infty) = (\Phi_{\vec{u},\vec{v},n,m})^{-1}(w_\infty)$, w has exactly k $\vec{P}$-runs of lengths $n_1, \ldots, n_k$, providing the colorings via the maps $\pi_{n_i,p,\vec{u}}$ for the corresponding P-runs in the monochromatic image w_∞ and similarly w has exactly

ℓ $\vec{x}$-runs of lengths $m_1, \ldots, m_\ell$, providing the colorings via the maps $\sigma_{m_j,q,\vec{v}}$ for the corresponding x-runs in w_∞. This yields the bijection:

$$\rho_{w_\infty} : (\Phi_{\vec{u},\vec{v},n,m})^{-1}(w_\infty) \longrightarrow \left(\prod_{i=1}^{k} C(n_i, p, \vec{u})\right) \times \left(\prod_{j=1}^{\ell} C(m_j, q, \vec{v})\right). \tag{50}$$

By Eq. (46), as w_∞ runs through the set $R_{\infty,\infty}(n, m; k, \ell)$, its triplet $(w_1, \vec{n}, \vec{m})$ runs through $R_{1,1}(k, \ell) \times G(n, k, \infty) \times G(m, \ell, \infty)$. Thus we have a bijection:

$$\begin{aligned} \rho : R_{\vec{u},\vec{v}}(n, m; k, \ell) &= (\Phi_{\vec{u},\vec{v},n,m})^{-1}(R_{\infty,\infty}(n, m; k, \ell)) \longleftrightarrow \\ R_{1,1}(k, \ell) \times \coprod_{\vec{n} \in G(n,k,\infty)} &\left(\prod_{i=1}^{k} C(n_i, p, \vec{u})\right) \times \coprod_{\vec{m} \in G(m,\ell,\infty)} \left(\prod_{j=1}^{\ell} C(m_j, q, \vec{v})\right) \end{aligned} \tag{51}$$

We gather below two results on coloring.

Lemma 5.1 *Let $\vec{v} = (\nu, \ldots, \nu)$ be a vector with q identical coordinates where ν is either an integer $\geqslant 1$ or ∞. Then*

$$c(b, q, \vec{v}) = \begin{cases} 1 & \text{if } q = 1 \text{ and } b \leqslant \nu; \\ 0 & \text{if } q = 1 \text{ and } b > \nu; \\ \sum_{\beta=1}^{b} g(b, \beta, \nu) q (q-1)^{\beta-1} & \text{if } q \geqslant 2. \end{cases} \tag{52}$$

If ν is infinite, then this simplifies to:

$$c(b, q, \vec{\infty}) = q^b. \tag{53}$$

Proof: The case when $q = 1$ is obvious. Suppose $q \geqslant 2$ and ν is finite (resp. infinite). The number β of same color runs in a coloring of b objects can be at most b. For each composition $b_1 + \cdot + b_\beta = b$ in $G(b, \beta, \nu)$, the run-lengths $b_1, \ldots, b_\beta$, which are uniformly bounded by ν (resp. unbounded), are fixed (therefore, the locations of the β runs are also fixed). The uniform bound (resp. unboundedness) allows any run be assigned any color, except that adjacent runs must have different colors. Thus there are $q(q-1)^{\beta-1}$ ways to choose the colors. Any two such colorings with distinct parameters are distinct. This proves the first statement.

Now suppose ν is infinite. Then we can color any of the b objects with any of the q colors and there are q^b ways. Alternatively, the second statement also follows from the first using Eq. (42) and the Binomial Theorem. $\square$

Corollary 5.2 *Let $\vec{v} = (\nu, \ldots, \nu)$ be the same as in* Lemma 5.1*. Then the generating series for $c(b, q, \vec{v})$ for fixed q and $\vec{v}$ is given by*

$$\mathbf{C}_{q,\vec{v}}(t) = \begin{cases} \mathbf{G}_{1,\nu}(t) & \text{if } q = 1; \\ \\ \dfrac{q\,\mathbf{G}_{1,\nu}(t)}{1-(q-1)\mathbf{G}_{1,\nu}(t)} & \text{if } q \geqslant 2. \end{cases}$$

If $\nu = \infty$, then this simplifies to

$$\mathbf{C}_{q,\vec{\infty}}(t) = \frac{qt}{1-qt}.$$

Proof: The case $q = 1$ is clear. For $q \geqslant 2$, the generating function is

$$\mathbf{C}_{q,\vec{v}}(t) = \sum_{b=1}^{\infty} \left(\sum_{\beta=1}^{b} g(b, \beta, \nu) q (q-1)^{\beta-1} \right) t^b$$

which, by Lemma 4.1 applied to

$$\mathbf{F}(t) = \sum_{\beta=1}^{\infty} q(q-1)^{\beta-1} t^{\beta} = \frac{qt}{1-(q-1)t},$$

is $\mathbf{F}(\mathbf{G}_{1,\nu}(t))$, as required. The $\nu = \infty$ case follows from Eqs. (53) and (43). □

We say the operators (resp. generators) have **uniform exponents** μ (resp. ν) if all the coordinates of the $\vec{P}$-exponent (resp. $\vec{x}$-exponent) vector u (resp. v) are equal to μ (resp. ν), in which case we write $\vec{u} = \vec{\mu}$ (resp. $\vec{v} = \vec{\nu}$). In the next two subsections, we consider the infinite exponent case, followed by the finite uniform exponent case.

5.2 Uniform exponents: Infinite cases

Theorem 5.3 *Let $R_{\vec{u},\vec{v}}$ be the set of Rota-Baxter words with p operators $\vec{P}$ and q generators $\vec{x}$, with corresponding uniform exponent vectors $\vec{u}$ and $\vec{v}$ respectively. Then for any positive integers n, m, we have*

(1) $r_{\vec{1},\vec{\infty}}(n, m) = q^m\, r_{1,\infty}(n, m)$,
(2) $r_{\vec{\infty},\vec{1}}(n, m) = p^n\, r_{\infty,1}(n, m)$,
(3) $r_{\vec{\infty},\vec{\infty}}(n, m) = p^n\, q^m\, r_{\infty,\infty}(n, m)$,

where $r_{1,\infty}(n, m)$, $r_{\infty,1}(n, m)$, and $r_{\infty,\infty}(n, m)$ are given by Corollary 4.3.

Proof: We will prove only (3). Let $w_\infty \in R_{\infty,\infty}(n,m;k,\ell)$. Each element $w \in R_{\vec{\infty},\vec{\infty}}(n,m;k,\ell)$ whose monochromatic image is w_∞ is obtained from w_∞ by coloring the P-runs and x-runs in w_∞. For any positive integer j, let $\mathbb{N}_j$ be the set consisting of the first j natural numbers. Since the number of mappings from $\mathbb{N}_n$ to $\mathbb{N}_p$ is p^n and the number of mappings from $\mathbb{N}_m$ to $\mathbb{N}_q$ is q^m, the result follows. Alternatively, this theorem also follows by summing over all k and ℓ the cardinalities from Eq. (51) and using Corollary 4.3. $\square$

Corollary 5.4 *The generating functions for the double sequences $r_{\vec{1},\vec{\infty}}(n,m)$, $r_{\vec{\infty},\vec{1}}(n,m)$, and $r_{\vec{\infty},\vec{\infty}}(n,m)$ are, respectively,*

(1) $\mathbf{R}_{\vec{1},\vec{\infty}}(z,t) = \mathbf{R}_{1,1}\left(z, \dfrac{qt}{1-qt}\right)$,

(2) $\mathbf{R}_{\vec{\infty},\vec{1}}(z,t) = \mathbf{R}_{1,1}\left(\dfrac{pz}{1-pz}, t\right)$, *and*

(3) $\mathbf{R}_{\vec{\infty},\vec{\infty}}(z,t) = \mathbf{R}_{1,1}\left(\dfrac{pz}{1-pz}, \dfrac{qt}{1-qt}\right)$.

Proof: By the above theorem and Corollary 4.5, for case (3), we have

$$\begin{aligned}
\mathbf{R}_{\vec{\infty},\vec{\infty}}(z,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} p^n q^m r_{\infty,\infty}(n,m) z^n t^m \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} r_{\infty,\infty}(n,m)(pz)^n (qt)^m \\
&= \mathbf{R}_{1,1}\left(\frac{pz}{1-pz}, \frac{qt}{1-qt}\right). \ \square
\end{aligned}$$

5.3 General case

Theorem 5.5 *Let $R_{\vec{u},\vec{v}}$ be the set of Rota-Baxter words with p operators $\vec{P}$ having exponent vector $\vec{u}$, and q generators $\vec{x}$ having exponent vector $\vec{v}$, where $\vec{u}$ (resp. $\vec{v}$) may have finite or infinite components. Let $\mathbf{C}_{p,\vec{u}}(z)$ (resp. $\mathbf{C}_{q,\vec{v}}(t)$) be the generating function for the sequence $c(a,p,\vec{u})$ as a varies (resp. $c(b,q,\vec{v})$ as b varies). Then the generating function for $r_{\vec{u},\vec{v}}(n,m)$ is*

$$\mathbf{R}_{\vec{u},\vec{v}}(z,t) = \mathbf{R}_{1,1}(\mathbf{C}_{p,\vec{u}}(z), \mathbf{C}_{q,\vec{v}}(t)).$$

Proof: The special cases when either $p = 1$, or $q = 1$, or both $p = q = 1$ specialize to corresponding cases of Theorem 4.4. Thus we shall assume $p \geqslant 2$ and $q \geqslant 2$. Some special cases when $\vec{u} = \vec{1}$ or $\vec{\infty}$ and when $\vec{v} = \vec{1}$ or $\vec{\infty}$ are included in Corollaries 4.5 and 5.4.

Consider the forgetful map $\Phi_{\vec{u},\vec{v},n,m}$ of Eq. (49), which is surjective:

$$\Phi_{\vec{u},\vec{v},n,m} : R_{\vec{u},\vec{v}}(n, m; k, \ell) \longrightarrow R_{\infty,\infty}(n, m; k, \ell)$$

We have obviously $R_{\vec{u},\vec{v}}(n, m; k, \ell) = (\Phi_{\vec{u},\vec{v},n,m})^{-1} (R_{\infty,\infty}(n, m; k, \ell))$, and using the bijection from Eq. (51), we have

$$\begin{aligned} & r_{\vec{u},\vec{v}}(n, m; k, \ell) \\ = & \, r_{1,1}(k, \ell) \left(\sum_{\vec{n} \in G(n,k,\infty)} \prod_{i=1}^{k} c(n_i, p, \vec{u}) \right) \left(\sum_{\vec{m} \in G(m,\ell,\infty)} \prod_{j=1}^{\ell} c(m_j, q, \vec{v}) \right) \end{aligned}$$

Recalling Eq. (44), we now calculate the generating function.

$$\begin{aligned} \mathbf{R}_{\vec{u},\vec{v}}(z, t) &= \sum_{n \geqslant 1} \sum_{m \geqslant 1} r_{\vec{u},\vec{v}}(n, m) z^n t^m \\ &= \sum_{n,m,k,\ell} r_{1,1}(k, \ell) \left(\sum_{\vec{n} \in G(n,k,\infty)} \prod_{i=1}^{k} c(n_i, p, \vec{u}) z^n \right) \left(\sum_{\vec{m} \in G(m,\ell,\infty)} \prod_{j=1}^{\ell} c(m_j, q, \vec{v}) t^m \right) \\ &= \sum_{k \geqslant 1} \sum_{\ell \geqslant 1} r_{1,1}(k, \ell) \left(\sum_{n \geqslant 1} \sum_{\vec{n}} \prod_{i=1}^{k} c(n_i, p, \vec{u}) z^n \right) \left(\sum_{m \geqslant 1} \sum_{\vec{m}} \prod_{j=1}^{\ell} c(m_j, q, \vec{v}) t^m \right) . \end{aligned}$$

Working on the last parenthesized expression, we have

$$\begin{aligned} \sum_{m \geqslant 1} \left[\sum_{\vec{m} \in G(m,\ell,\infty)} \prod_{j=1}^{\ell} c(m_j, q, \vec{v}) \right] t^m &= \sum_{m \geqslant 1} \left[\sum_{\vec{m} \in G(m,\ell,\infty)} \prod_{j=1}^{\ell} c(m_j, q, \vec{v}) t^{m_j} \right] \\ &= \left(\sum_{b \geqslant 1} c(b, q, \vec{v}) t^b \right)^{\ell} \\ &= \left(\mathbf{C}_{q,\vec{v}}(t) \right)^{\ell} . \end{aligned}$$

The result now follows from the definition of $\mathbf{R}_{1,1}(z, t)$. $\square$

Corollary 5.6 *If the exponent vectors $\vec{u}, \vec{v}$ are uniform, say $\vec{u} = \vec{\mu}$ and $\vec{v} = \vec{\nu}$ for some finite or infinite constants μ, ν, then the generating function for $r_{\vec{\mu},\vec{\nu}}(n, m)$ is:*

$$\mathbf{R}_{\vec{\mu},\vec{\nu}}(z, t) = \mathbf{R}_{1,1}\left(\frac{p\,\mathbf{G}_{1,\mu}(z)}{1-(p-1)\mathbf{G}_{1,\mu}(z)}, \frac{q\,\mathbf{G}_{1,\nu}(t)}{1-(q-1)\mathbf{G}_{1,\nu}(t)}\right).$$

Proof: This follows from Lemma 5.1 and Corollary 5.2. □

6 Conclusion and outlook

We have obtained generating functions and algorithms related to enumeration of sets of Rota-Baxter words in various generalities. It is interesting to see the close relation with Catalan numbers, further revealing the combinatorial nature of Rota-Baxter algebras in cases where the sets form canonical bases.

The cases we have considered allow multiple operators and multiple generators with uniform exponents. The case where the operators and generators are allowed to have variable exponents are in principle solved and it only remains to compute the generating functions for the number of colorings. Since the coloring problem is a constraint satisfaction problem, we are investigating this connection to see if the result is alrady known. We expect these results to be useful in the construction of more general Rota-Baxter type algebras, where the unary operators may satisfy other identities instead.

We mentioned a few new integer sequences in this study. Clearly, since the most general generating functions given by Theorem 5.5 are parametrized by two arbitrary vectors, we expect many specialized sequences to be unknown in the Sloane database. The connections between sets of Rota-Baxter words and other sets of combinatorial objects when their counting sequences are the same will also be of tremendous interest in expanding the realm of symbolic computation because such connections will allow the representation of these other sets of combinatorial objects using Rota-Baxter words which are purely algebraic in nature.

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